$$-\frac{7}{3}e^{\frac{\pi}{n}}\sin(\zeta+2\tau+\xi)$$
$$+\frac{15}{16}e^{\frac{\pi}{n}}\sin(\zeta-4\tau+\xi)\,.$$

[TO BE CONTINUED.]

NOTE ON A UNIQUE PROPERTY OF AN AXISYMMETRIC DETERMINANT OF THE FOURTH ORDER.

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The determinant in question is

$$\begin{vmatrix} 0 & a & b & c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & k \end{vmatrix}, \text{ or } \Omega \text{ say,}$$

its only specialty as an axisymmetric determinant of the fourth order being that it has a zero in the place (1,1). Now if we take the minor

$$\begin{vmatrix} 0 & a & b \\ a & d & e \\ b & e & g \end{vmatrix}, \text{ or } K \text{ say,}$$

and substitute in it for a and b the complementary minors of h and f in Ω, it is found that, curiously enough, the new determinant is divisible by the old, the quotient being

$$\begin{vmatrix} 0 & a & b & -c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & 0 \end{vmatrix}$$

That is to say, if H and F denote the complementary minors of h and f in Ω we assert the identity

$$\begin{vmatrix} 0 & H & F \\ H & d & e \\ F & e & g \end{vmatrix} = \begin{vmatrix} 0 & a & b \\ a & d & e \\ b & e & g \end{vmatrix} \cdot \begin{vmatrix} 0 & a & b & -c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & 0 \end{vmatrix}.$$

This may be established in various ways, but perhaps the following is as neat as any other:—

Multiplying the left-hand member column-wise by

$$\begin{vmatrix} 1 & 0 & 0 \\ bc & 1 & 0 \\ -ac & 0 & 1 \end{vmatrix}$$

and bearing in mind that

$$H = \begin{vmatrix} 0 & a & b \\ a & d & e \\ c & f & h \end{vmatrix} = abf + ace - bcd - a^2h,$$

and

$$F = \begin{vmatrix} 0 & a & b \\ b & e & g \\ c & f & h \end{vmatrix} = b^2f + acg - bce - abh,$$

we have

$$\begin{vmatrix} 0 & H & F \\ H & d & e \\ F & e & g \end{vmatrix} = \begin{vmatrix} bcH - acF & a(bf - ah) & b(bf - ah) \\ H & d & e \\ F & e & g \end{vmatrix}.$$

Multiplying again by the same multiplier, but now performing the operation row-wise in order that the determinant may re-assume the original symmetrical form, we have

$$\begin{vmatrix} 0 & H & F \\ H & d & e \\ F & e & g \end{vmatrix} = \begin{vmatrix} bcH - acF & a(bf - ah) & b(bf - ah) \\ a(bf - ah) & d & e \\ b(bf - ah) & e & g \end{vmatrix}.$$

But since

$$\begin{vmatrix} 0 & a & b & 0 \\ a & d & e & a \\ b & e & g & b \\ c & f & h & c \end{vmatrix} = 0,$$

it follows that

$$cK - bH + aF = 0,$$

and thus

$$bcH - acF = c^2K.$$

Substituting this in our last obtained result, and partitioning the determinant into two, we have

$$\begin{vmatrix} c^2K & 0 & 0 \\ 0 & d & e \\ 0 & e & g \end{vmatrix} + \begin{vmatrix} 0 & a(bf-ah) & b(bf-ah) \\ a(bf-ah) & d & e \\ b(bf-ah) & e & g \end{vmatrix},$$

i. e.,

$$c^2K(dg-e^2)+(bf-ah)^2K,$$

i. e.,

$$K\,[c^2(dg-e^2)+(bf-ah)^2]\,,$$

i. e.,

$$K\begin{vmatrix} 0 & a & b & -c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & 0 \end{vmatrix},$$

as was to be proved.

In connection with the co-factor of K here it is worth while remarking upon the very considerable cancelling effect of altering the sign of one of the non-axial elements in an axisymmetric determinant. If the determinant be of the n^{th} order the number of terms is of course $n!$, or classifying them as they are got when we expand the determinant according to binary products of elements from the first row and first column, we may say that the number of terms is

$$(n-1)!+(n-2)!\,(n-1)+2C_{n-1,2}\,(n-2)!\,.$$

Now if one of the non-axial elements be altered in sign, say the element in the place $(1, n)$, the number of terms obtained in like manner is

$$(n-1)!+(n-2)!\,(n-1)+2C_{n-2,2}\,(n-2)!\,.$$

The number in the latter case is thus diminished by

$$2\,(n-2)!\,(C_{n-1,2}-C_{n-2,2})$$

i. e.,

$$2\,(n-2)!\,(n-2)\,.$$